ON THE HARMONIC SUMMABILITY OF LAGUERRE SERIES

BY

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ABSTRACT

In the present paper we discuss the harmonic summability of Laguerre series associated with a Lebesgue-measurable function at the frontier point $x = 0$.

1. Let $f(x)$ be a Lebesgue-measurable function such that the integral

$$\int_0^\infty e^{-x} x^\alpha f(x) L_n^{(\alpha)}(x)\, dx, \qquad \alpha > -1$$

exists, where $L_n^{(\alpha)}(x)$ denotes the nth Laguerre polynomial of order α.

The Laguerre series corresponding to this function $f(x)$ is

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \tag{1.1}$$

in which

$$a_n = \left\{ \Gamma(\alpha+1) A_n^\alpha \right\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) L_n^{(\alpha)}(y)\, dy. \tag{1.2}$$

and

$$A_n = \binom{n+\alpha}{n} \sim n^\alpha. \tag{1.3}$$

A sequence $\{S_n\}$ is said to be summable by harmonic means, if

$$\lim_{n\to\infty} (\log n)^{-1} \sum_{k=0}^{n} \frac{S_{n-k}}{k+1}$$

exists.

2. The object of this paper is to investigate the harmonic summability of Laguerre series (for certain values of α) at the frontier point $x = 0$.

We write

$$\phi(y) = \{\Gamma(\alpha+1)\}^{-1} e^{-y} [f(y) - A] y^\alpha.$$

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We establish the following theorem:

THEOREM. *For* $-\frac{1}{2} \geqq \alpha \geqq -\frac{5}{6}$, *the series* (1.1) *is summable by harmonic means at the point* $x = 0$ *to the sum* A, *provided,*

$$\Phi(t) \equiv \int_0^t |\phi(y)|\,dy = o(t^{\alpha+1}), \qquad \text{as } t \to 0. \tag{2.1}$$

$$\int_\omega^n e^{y/2} y^{-\alpha/2-3/4} |\phi(y)|\,dy = o(n^{-\alpha/2-1/4}) \tag{2.2}$$

and

$$\int_n^\infty e^{y/2} y^{-1/3} |\phi(y)|\,dy = o(1). \tag{2.3}$$

3. We require the following lemmas in the proof of our theorem:

LEMMA 1. [1, p. 175]. *Let* α *be arbitrary and real,* c *and* ω *fixed positive constants,* $n \to \infty$, *then*

$$L_n^{(\alpha)}(x) = \begin{cases} x^{-\alpha/2-1/4} O(n^{\alpha/2-1/4}), & \frac{c}{n} \leqq x \leqq \omega; \\ O(n^\alpha), & 0 \leqq x \leqq c/n. \end{cases} \tag{3.1}$$

LEMMA 2. [1, p. 238]. *If* α *be real and arbitrary,* $\omega > 0$, $0 < \eta < 4$, *then for* $n \to \infty$, *we have*

$$\max e^{-x/2} x^{\alpha/2+1/4} |L_n^{(\alpha)}(x)| = \begin{cases} n^{\alpha/2-1/4}, & \omega \leqq x \leqq (4-\eta)n; \\ n^{\alpha/2-1/12}, & x \geqq \omega. \end{cases} \tag{3.2}$$

4. Proof of the theorem. Let S_n denote the nth partial sum of the series (1.1) at the point $x = 0$, then

$$\begin{aligned} S_n &= \left\{\Gamma(\alpha+1)\right\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) \sum_{m=0}^n L_m^{(\alpha)}(y)\,dy, \\ &= \left\{\Gamma(\alpha+1)\right\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y)\, L_n^{(\alpha+1)}(y)\,dy, \end{aligned} \tag{4.1}$$

Thus, by the definition

$$(4.2)\qquad (\log n)^{-1}\sum_{k=0}^{n}\{S_{n-k}-A\}$$

$$= (\log n)^{-1}\sum_{k=0}^{n}\frac{1}{k+1}\left\{\sqrt{(\alpha+1)}\right\}^{-1}\int_0^\infty e^{-y}y^\alpha[f(y)-A]\,L_{n-k}^{(\alpha+1)}(y)dy$$

$$= (\log n)^{-1}\int_0^\infty \phi(y)\sum_{k=0}^{n}\frac{1}{k+1}L_{n-k}^{(\alpha+1)}(y)dy\,.$$

$$(4.3)\qquad = (\log n)^{-1}\left[\int_0^{c/n}+\int_{c/n}^{\omega}+\int_{\omega}^{n}+\int_n^\infty\right]\phi(y)\sum_{k=0}^{n}\frac{1}{k+1}L_{n-k}^{(\alpha+1)}(y)dy$$

$$(4.4)\qquad = A+B+C+D,$$

say. Now by the help of Lemma 1, we have

$$(4.5)\qquad A = (\log n)^{-1}\sum_{k=0}^{n}\frac{1}{k+1}\int_0^{c/n}|\phi(y)|\,O(n-k)^{\alpha+1}dy$$

$$= (\log n)^{-1}O(n^{\alpha+1}\log n)\int_0^{c/n}|\phi(y)|\,dy$$

$$(4.6)\qquad = o(1),$$

by the hypothesis (2.1).

Again, using Lemma 1, we get

$$(4.7)\qquad B = (\log n)^{-1}\sum_{k=0}^{n}\frac{O(n-k)^{(\alpha+1)/2-1/4}}{k+1}\int_{c/n}^{\omega}|\phi(y)|\,y^{-(\alpha+1)/2-1/4}dy$$

$$= (\log n)^{-1}\sum_{k=0}^{n}\frac{O(n-k)^{\alpha/2+5/4}}{(n-k)(k+1)}\int_{c/n}^{\omega}|\phi(y)|\,y^{-\alpha/2-3/4}\,dy$$

$$= (\log n)^{-1}.\,O(n^{\alpha/2+5/4}).\,O(n^{-1}\log n).$$

$$\cdot\left[\left\{y^{-\alpha/2-3/4}\Phi(y)\right\}_{c/n}^{\omega}+\int_{c/n}^{\omega}\Phi(y)y^{-\alpha/2-5/4}dy\right],$$

ntegrating by parts.

$$= o(1)+o(n^{\alpha/2+1/4})\int_{c/n}^{\omega}y^{\alpha/2-3/4}dy\,.$$

by the hypothesis (2.1).

$$(4.8)\qquad = o(1).$$

Now, using Lemma 2, we get

$$(4.9)\qquad C = (\log n)^{-1}\int_{\omega}^{n}|\phi(y)|\sum_{k=0}^{n}\frac{1}{k+1}e^{y/2}(n-k)^{(\alpha+1)/2-1/4}y^{-(\alpha+1)/2-\frac{1}{4}}dy$$

$$= (\log n)^{-1}\int_{\omega}^{n}e^{y/2}y^{-(\alpha/2)/-3/4}|\phi(y)|\sum_{k=0}^{n}\frac{O(n-k)^{\alpha/2+5/4}}{(n-k)(k+1)}\,dy$$

$$= (\log n)^{-1}O\left[n^{\alpha/2+5/4}\sum_{k=0}^{n}\frac{1}{(n-k)(k+1)}\int_{\omega}^{n}e^{y/2}y^{-\alpha/2-3/4}|\phi(y)|dy\right.$$

$$= (\log n)^{-1}O(n^{\alpha/2+5/4})\,O(n^{-1}\log n)\,o(n^{-\alpha/2-1/4}),$$

by the hypothesis (2.2).

$$(4.10)\qquad = o(1).$$

Finally, by the second condition of Lemma 2, we find

$$(4.11)\quad D = (\log n)^{-1}\int_{n}^{\infty}|\phi(y)|\sum_{k=0}^{n}\frac{1}{k+1}(n-k)^{(\alpha+1)/2-1/12}y^{-(\alpha+1)/2-1/4}e^{y/2}dy$$

$$= (\log n)^{-1}O(n^{\alpha/2+5/12})\sum_{k=0}^{n}\frac{1}{k+1}\int_{n}^{\infty}\frac{e^{y/2}y^{-1/3}|\phi(y)|}{y^{\alpha/2+5/12}}dy$$

$$= (\log n)^{-1}O(n^{\alpha/2+5/12}\log n)n^{-\alpha/2-5/12}\int_{n}^{\infty}e^{y/2}y^{-1/3}|\phi(y)|\,dy_e$$

$$(4.12)\qquad = o(1),$$

by the hypothesis (2.3).

This demonstrates the theorem.

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Reference

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